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ON THE GREATEST COMMON DIVISOR AND HIGHEST COMMON MULTIPLE OF TWO INTEGERS

SHUAA ALDHAFEERI
and MASHHOUR BANI-ATA

Department of Mathematics
College of Basic Education
PAAET
Kuwait
e-mail: saldhafeeri@yahoo.com
mashhour_ibrahim@yahoo.com

Abstract

The aim of this note is to answer the following two problems:

Problem 1: Given two positive integers $m, n,$ such that $m|n$. What are the pairs (a, b) of positive integers, such that $m = \gcd(a, b)$ and $n = \operatorname{lcm}(a, b)$? where $\gcd(a, b)$ denotes the greatest common divisor of a and b; and $\operatorname{lcm}(a, b)$ denotes the least common multiple of a and b.

Problem 2: How one can construct the pairs (a, b) with the above properties, i.e., $m = \gcd(a, b)$ and $n = \operatorname{lcm}(a, b)$?

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1. Introduction

The fundamental theorem of arithmetic is an important result that shows that the primes are building blocks of the integers. Here is the statement of the theorem: Every positive integer greater than one can be written uniquely as a product of primes [1]. The proofs of the results in this paper are mainly based on this theorem.

Proof of Problem 1. The number of such pairs is 2^{k-1}, where k is the number of prime divisors of $\frac{n}{m}$. This can be shown as follows:

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ be the prime divisors of n / m, for distinct primes $p_1, p_2, \cdots, p_s$ with exponents $e_1, e_2, \cdots, e_s$, which are all positive. As m divided n, we have $m = p_1^{f_1} p_2^{f_2} \cdots p_s^{f_s}$, where $0 \leq f_i \leq e_i$.

Then $m = \gcd(a, b)$ and $n = \operatorname{lcm}(a, b)$, if and only if the following holds: $a = p_1^{f_1 + x_1} p_2^{f_2 + x_2} \cdots p_s^{f_s + x_s}$ and $b = p_1^{f_1 + y_1} p_2^{f_2 + y_2} \cdots p_s^{f_s + y_s}$, where $0 \leq x_i, y_i \leq e_i - f_i$ for $i = 1, 2, \cdots, s$.

Moreover, for each i, it must hold that $(x_i, y_i) = (0, e_i - f_i)$ or $(x_i, y_i) = (e_i - f_i, 0)$. Hence for each i with $f_i < e_i$, there are two choices for the pairs (x_i, y_i). The number of i, s with $f_i < e_i$ is equal to the number k of distinct prime divisors of $\frac{n}{m}$. Hence the claim. But if we do not make a distinction between the pairs (a, b) and (b, a), then we can divide by 2 and hence the number of distinct pairs is $\frac{2^k}{2} = 2^{k-1}$.

Example 1. Let $m = 30 = 2 \cdot 3 \cdot 5$ and $n = 450 = 2 \cdot 3^2 \cdot 5^2$, then $\frac{n}{m} = 3 \cdot 5$ has two distinct prime divisors.

Hence, there are 2 admissible pairs $(a, b) = (2 \cdot 3 \cdot 5,\ 2 \cdot 3^2 \cdot 5^2)$ or $(a, b) = (2 \cdot 3 \cdot 5^2,\ 2 \cdot 3^2 \cdot 5)$, with $30 = \gcd(a, b)$ and $450 = \operatorname{lcm}(a, b)$.

Proof of Problem 2. Let $\frac{n}{m} = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be the prime decomposition of the quotient $\frac{n}{m}$. Set $q_i = p_i^{e_i}$. Then the admissible pairs (a, b) with $m = \gcd(a, b)$ and $n = \operatorname{lcm}(a, b)$ are precisely the pairs:

$a = m \cdot \prod_{i \in S} q_i$, where $\prod_{i \in S} q_i$ denotes the product of $\{q_i \mid i \in S\}$, and

$b = m \cdot \prod_{j \in S} q_j$ for subsets S of $\{1, 2, \cdots, k\}$.

There are 2^k such subset S, and hence 2^k such pairs (a, b). If we do not make a distinction between the pairs (a, b) and (b, a), then we have 2^{k-1} distinct pairs.

Example 2. Let $m = 18 = 2 \cdot 3^2$ and $n = 540 = 2^2 \cdot 3^3 \cdot 5$. Hence $\frac{n}{m} = 2 \cdot 3 \cdot 5$ with $q_1 = 2$, $q_2 = 3$, and $q_3 = 5$. So, we get 2^3 pairs (a, b) with $18 = \gcd(a, b)$ and $540 = \operatorname{lcm}(a, b)$, and they are listed below:

$S = \{\ \}$ $\quad (a, b) = (18, 540)$,

$S = \{1\}$ $\quad (a, b) = (36, 270)$,

$S = \{1, 2\}$ $\quad (a, b) = (108, 90)$,

$S = \{1, 2, 3\}$ $\quad (a, b) = (540, 18)$,

$S = \{1, 3\}$ $\quad (a, b) = (180, 54)$,

$S = \{2\}$ $\quad (a, b) = (54, 180)$,

$$S = \{2, 3\} \qquad (a, b) = (270, 36),$$

$$S = \{3\} \qquad (a, b) = (90, 108).$$

Reference

[1] K. H. Rosen, Elementary Number Theory and its Applications, 2nd Edition, Addison, Wesley Publication Company, 1988.

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